Generalising Homogeneous Structures

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Contents

Abstract

The purpose of my project is two-fold. It will aim to first summarise and express some of the work currently being undertaken in the field, before moving on to a novel exploration which aims to generalise previous work in the study of homomorphism-homogeneous graphs.

I will begin by outlining some of the necessary underpinnings for the project, defining some key terms for graphs, k-hypergraphs, relational structures, and homogeneity. After proving Fraïssé's Theorem and briefly outlining the state of research on homogeneous graphs and k-hypegraphs, I begin to present my original work.

The original work I present includes extending the well known Rado Graph to k-hypergraphs, proving some general properties of some homomorphism-homogeneous k-hypergraphs, and finishes with a proof that there are uncountably many of a certain kind of homomorphism-homogeneous k-hypergraph, for any $k \geq 3$.

I certify that this project report has been written by me, is a record of work carried out by me, and is essentially different from work undertaken for any other purpose or assessment.

Chapter One

Introduction

In brief, homogeneous structures can be considered as the "most symmetric" structures possible. This has made them an interesting focus of investigation both for their own sake, but also for their significance to many other areas of mathematics. They arise from the intersection of many disparate areas of discrete mathematics, from model theory and combinatorics to permutation group theory.

They were first studied by Fraïssé in the early 1950's ([11]), who's eponymous Amalgamation Theorem provided a huge leap forward in the classification of homogeneous structures. It meant that examples of homogeneous structures could be found by providing amalgamation classes of finite structures, since it guarantees that all homogeneous structures arise from these.

In the ensuing decades, there were various efforts to use this framework to begin to classify various homogeneous structures. These led to success in classifying homogeneous graphs by Woodrow and Lachlan in 1980 [10], and inspired the search for more exotic homogeneous structures such as k-hypergraphs, highlighted by the work of Akhtar and Lachlan in 1995 [1].

One exciting avenue of research which began in earnest with Cameron and Nešetřil's work [5] in the early 2000's was that of homomorphism-homogeneity. This opened the door to a whole new class of structures to study, analyse, and attempt to classify. These attempts are still proving fruitful to this day: in 2017, Coleman's PhD Thesis [6] investigated analogues for Fraïssé's Theorem in this context, and found uncountably many examples of certain kinds of homomorphism-homogeneous graphs.

In my thesis, I hope to flesh out some of the above developments in more detail, but the focus is on continuing in the line of the above research to further study and classify homomorphism-homogeneous structures, with a focus on k-hypergraphs.

The target audience for this project are mathematicians with at least as much experience as an advanced undergraduate. The aim for the project is to provide a relatively selfcontained description of my research, giving an understanding of the necessary preliminaries before outlining my own contributions. It will hopefully be of interest for those new to the field, as well as those already in the field who are looking to understand my own contributions.

For a degree of brevity, this thesis assumes some familiarity with graph theory, although many definitions will be covered in the introduction as we introduce terminology associated with k-hypergraphs.

The project will serve to illuminate some of the key challenges faced when working in this area, and will provide some open questions for others looking to investigate this area further.

1.1 Basic Graph and k-Hypergraph Theory

Before we look at homogeneity in its general context, we will define some terminology we will use throughout relating to graphs and k-hypergraphs.

1.1.1 Graphs

Definition 1.1. A *graph* is an ordered pair $G = (V,E)$ where V is a set of *points* or *vertices*, and $E \subset P(V)$, such that E is a set of pairs of points, known as *edges*.

If $e \in P(V)$, $|e| = 2$ but $e \notin E$, then e is a *non-edge* of G.

Despite this combinatorial description, graphs are easily represented pictorially, as seen in Figure 1.1

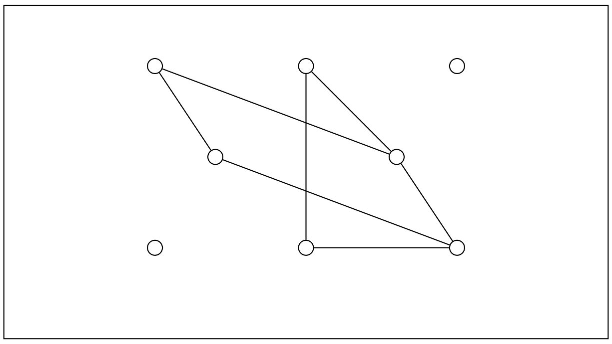


Figure 1.1: Pictorial representation of a graph. Points are represented by circles, and edges by lines between points.

We will introduce some terminology here which will be used throughout the paper, see Diestel [8] for examples and explanations of the following.

Definition 1.2. A *path* is a non-empty graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, ..., x_k\}, E = \{x_0x_1, x_1x_2, ..., x_{k-1}x_k\},$$

where the x_i are all distinct.

Figure 1.2: An example of a path on 5 vertices.

Definition 1.3. A non-empty graph is called *connected* if, for any two vertices u and v in G, there exists a path from u to v in G.

Definition 1.4. The *distance* $d_G(x, y)$ in G of two vertices x, y is the length of a shortest path between x and y in G.

Definition 1.5. The *diameter* is the greatest distance between any two vertices in the graph G.

Definition 1.6. A *cycle* is a non-empty graph $C = (V, E)$ of the form

$$V = \{x_0, x_1, ..., x_k\}, E = \{x_0x_1, x_1x_2, ..., x_{k-1}x_k, x_kx_0\},$$

where the x_i are all distinct.

Note how a cycle is simply a path with an edge between the first and last point.

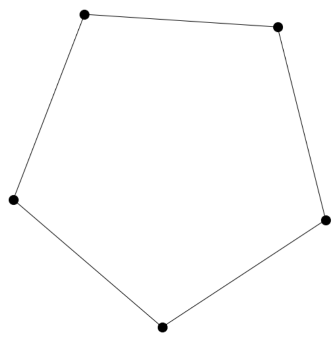


Figure 1.3: An example of a cycle on 5 vertices.

Definition 1.7. Let $G = (V, E)$ be a graph and let K consist of all 2-element subsets of V. Then $\bar{G} = (V,\, K \setminus E)$ is the *complement* of G.

Definition 1.8. Given a set of vertices V, the *complete graph* K_V is the graph such that there is an edge between every pair of distinct points in V. The *null graph* on vertices V is the complement of K_V.

Definition 1.9. If $G = (V, E)$ and $G' = (V', E')$ are graphs such that $V' \subseteq V$ and $E' \subseteq E$, then G' is a *subgraph* of G. If, furthermore, G' contains all the edges $(x, y) \in E$ where $x, y \in V'$, then G' is an *induced subgraph* of G. We may also refer to this as the *induced graph on* V'.

1.1.2 k-Hypergraphs

Definition 1.10. A *k-hypergraph* Γ is an ordered pair (V, E) where V is a set of points, and $E \subset P(V)$ a set of edges or *k-hyperedges* such that for each $e \in E$, we have $|e| = k$.

If $e \in P(V)$, $|e| = k$ but $e \notin E$, then e is a *non-edge* of Γ.

The intuition behind this is that instead of considering relations between two points in a graph, we can consider relations between any k points. For $k > 2$ these are harder to visualise than graphs, but it is still possible by viewing collections of points as edges, as shown in Figure 1.4

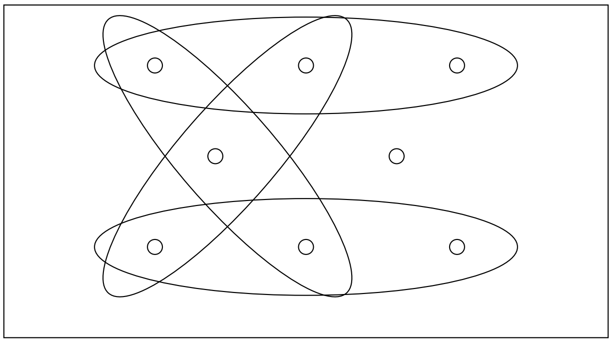


Figure 1.4: Pictorial representation of a 3-hypergraph. Each circle represents a point, each ellipse an edge.

Just as we did for graphs, we will provide some definitions for some frequently used concepts in k-hypergraphs. Unlike the definitions for graphs, these definitions are non-standard.

Definition 1.11. Given a k-hypergraph $\Gamma = (V, E)$, and a point of V denoted by v, we define its *neighbourhood* to be the set of subsets of size $k - 1$ that form an edge with v.

Definition 1.12. A *path from u to v* for vertices u and v in V is a non-empty k-hypergraph $P = (V, E)$ of the form

$$V = \{x_0 = u, x_1, ..., x_n = v\}, E = \{e_0, e_1, ..., e_k\},$$

where $e_i \cap e_j \neq \emptyset$ for all $0 \leq i \leq k - 1$, and $u \in e_0, v \in e_k$

Remark 1.13. *Notice how a path in the graph sense refers to a specific graph, but the same doesn't hold in general for k-hypergraphs. There can be several non-isomorphic paths on vertices $v_0, ..., v_n$.*

Definition 1.14. A non-empty k-hypergraph Γ is called *connected* if, for any two vertices u and v in Γ, there exists a path from u to v in Γ.

(a) Path 1 (b) Path 2

Figure 1.5: Two non-isomorphic paths in a 3-hypergraph from vertex 1 to vertex 7, both on seven vertices.

Definition 1.15. The *distance* $d_\Gamma(x, y)$ in Γ of two vertices x, y is the length of a shortest path between x and y in Γ.

Definition 1.16. The *diameter* is the greatest distance between any two vertices in the k-hypergraph Γ.

Remark 1.17. *Although the definition is analogous to that of diameter in graphs, having low diameter is in many senses a much weaker property for k-hypergraphs than it is for graphs. See Figure 1.6 for an example of this.*

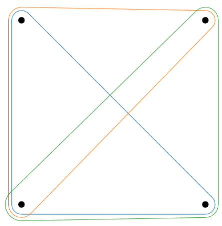


Figure 1.6: A 3-hypergraph with diameter 1, but that isn't complete (there's no edge between the top vertices, and the bottom right one).

Now, in light of Remark 1.13, there is no single k-hypergraph that we can define as a path on some vertices, and hence there is no single object that we can refer to as a cycle on some vertices. However, the following definition is the most natural for our purposes.

Definition 1.18. A (k, n)-*cycle* where $n > k$ is a non-empty k-hypergraph $C_n^k = (V, E)$ of the form

$$V = \{x_0, x_1, ..., x_{n-1}\}, E = \{\{x_0, x_1, x_2, ..., x_{k-1}\}, \{x_1, x_2, x_3, ..., x_k\}, , ..., \{x_{n-k}, x_{n-k+1}, ..., x_{n-1}\},$$
$$\{x_{n-k+1}, x_{n-k+2}, ..., x_{n-1}, x_0\}, ..., \{x_{n-1}, x_0, x_1, ..., x_{k-2}\}\}$$

where the x_i are all distinct.

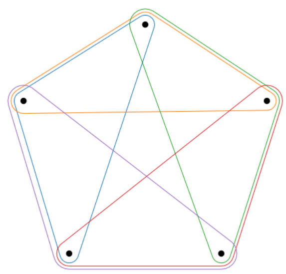
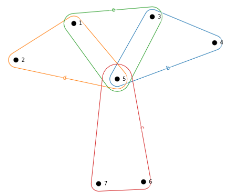


Figure 1.7: An example of a (3,5)-cycle, a 3-hypergraph on 5 vertices that generalises the notion of a cycle on a graph.

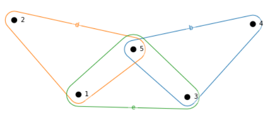
Definition 1.19. Let $\Gamma = (V, E)$ be a k-hypergraph and let K consist of all k-element subsets of V. Then $\bar{\Gamma} = (V, K \setminus E)$ is the *complement* of Γ.

Definition 1.20. Given a set of vertices V, the *complete k-hypergraph* K_V^k is the k-hypergraph such that there is an edge between any k distinct points in V. The *null k-hypergraph* on vertices V is the complement of K_V^k.

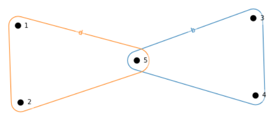
Definition 1.21. If $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ are k-hypergraphs such that $V' \subseteq V$ and $E' \subseteq E$, then Γ' is a *sub k-hypergraph* of Γ. For ease, we may refer to this as a *sub-hypergraph*. If, furthermore, Γ' contains all the edges $\{x_1, ..., x_k\} \in E$ such that $x_1, ..., x_k \in V'$, then Γ' is an *induced sub k-hypergraph* of Γ. We may also refer to this as the *induced k-hypergraph on V'*.



(a) A 3-hypergraph Γ



(b) An induced 3-hypergraph of Γ on the vertices $\{1, 2, 3, 4, 5\}$



(c) A sub 3-hypergraph of Γ.

Figure 1.8: An example 3-hypergraph, and two of its sub 3-hypergraphs. Notice that (b) is an induced sub k-hypergraph as it contains all of the edges between the points in $\{1, 2, 3, 4, 5\}$, but (c) is not, as it is missing the edge $\{1, 3, 5\}$.

1.2 Relational Structures

Although we could just define homogeneity in terms of graphs and k-hypergraphs, we will give its more general definition here, in terms of relational structures.

Definition 1.22. A relational structure M is homogeneous if whenever U, V are isomorphic finite substructures of M, we can extend this isomorphism to an automorphism of M.

Remark 1.23. *Although a homogeneous structure can be of any cardinality, unless stated otherwise assume all homogeneous, and later homomorphism-homogeneous structures, are countably infinite.*

Firstly, what is a relational structure? This requires a detour into model theory, with a technical definition. For further information on Model Theory, see Hodges [9].

Definition 1.24. A *relational $\mathcal{L}$-Structure* $\mathcal{A}$ is a triple (A, $\mathcal{L}$, I), where:

- A is a set, referred to as the *domain*.
- $\mathcal{L}$ is the *language* or the *signature* of the structure, which consists of a set $L = \{R_i | i \in I\}$ (for some index set I) of relation symbols along with a function $arity : L \to \mathbb{N}_0$ which gives the arity of the interpretation of each relation in L, meaning how many elements the relation is defined on. We will describe a relation between n elements as *n-ary*.
- $I = \{R_i^A | i \in I\}$ is the *interpretation function*, which consists of *interpretations* which assign relations to each relation symbol in the signature.

Remark 1.25. *This definition seems complex, but a good intuition is the following: the domain is the underlying set we define our structure on; the signature gives which kind of structure it is; and the interpretation function gives us the specific structure we have. This means that we can meaningfully compare any structures with the same language, as they are "different instances of the same kind of structure".*

We also define substructures, which we will use in the proof of Fraïssé's Theorem.

Definition 1.26. Given two structures A and B with the same language $\mathcal{L}$, A is a *substructure* of B if the domain of A is a subset of the domain of B, and $R_i^A = R_i^B \cap A^n$ for every n-ary relation symbol R_i in $\mathcal{L}$. Denote this by $A \leq B$.

Example 1.27. Consider a graph $G = (V, E)$. We can view this as a relational $\mathcal{L}$-structure, where the domain is the set of vertices V, and the language $\mathcal{L}$ is given by a single binary relation R_E. Then, define the interpretation R_E^G such that $(x, y) \in R_E^G$ if and only if $\{x, y\}$

is an edge in G.

Similarly we can consider a k-hypergraph Γ as a $\mathcal{L}_k$-structure, where the domain is a set of vertices V, and the language $\mathcal{L}_k$ is given by a single k-ary relation R_E. Then, define the interpretation R_E^Γ such that $(x_1, ..., x_k) \in R_E^\Gamma$ if and only if $\{x_1, ..., x_k\}$ is an edge in Γ.

This has the following important consequence: **all definitions and theorems for relational structures apply to graphs and k-hypergraphs.**

Definition 1.28. Let $\mathcal{A}, \mathcal{B}$ be relational $\mathcal{L}$-structures, with domains A and B respectively and suppose $f : A \to B$. Then, a function $f : \mathcal{A} \to \mathcal{B}$ is a *homomorphism* if for all $R_i \in \mathcal{L}$, and for any $a_1, ..., a_n \in A$, we have that $(a_1, ..., a_n) \in R_i^A$ implies $(f(a_1), ..., f(a_n)) \in R_i^B$ (where $arity(R_i) = n$).

Remark 1.29. *From Definition 1.28 we can see that homomorphisms preserve relations, but they don't necessarily preserve non-relations. For instance, in the case of graphs, there is no relation for non-edges, so homomorphisms may map non-edges to edges.*

1.2.1 Partial and Surjective Maps

To lay the groundwork for working on homomorphism-homogeneity, we will define some common maps between relational structures. These work for any relational structures, but we will use them mainly in the context of graphs and k-hypergraphs.

We will label some of these, in the vein of Coleman's work on homomorphism-homogeneous structures [6], to allow easier reference to them.

Let X, Y be sets, and let $f : X \to Y$ be a function. Then:

Definition 1.30. f is *injective* if for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Definition 1.31. f is *surjective* if for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

Now, we can combine the above definitions with homogeneous maps on relational structures A and B. We will label homomorphisms with (H). Firstly, we look at definitions relating to injectivity:

Definition 1.32. $f : A \to B$ is a *monomorphism* if f is an injective homomorphism (M).

Definition 1.33. If $f : A \to B$ is a monomorphism that preserves non-relations, then f is an *embedding.*

Definition 1.34. If $f : A \to B$ is a bijective embedding, then it is an *isomorphism* (I).

These are the maps that we will consider when looking at *partial maps* in structures, which we will refer to as maps that don't have the entirety of the structure M in their domain of definition.

However, to talk about the *global maps*, which we will refer to as maps from the entire structure M to itself, we will need further terminology.

Definition 1.35. If $f : M \to M$ is a homomorphism, it is an *endomorphism* of M (H).

If $f : M \to M$ is a surjective endomorphism, call it an *epimorphism* of M (E).

If $f : M \to M$ is an injective endomorphism, call it a *monomorphism* of M (M).

If $f : M \to M$ is a bijective endomorphism, call it a *bimorphism* of M (B).

Remark 1.36. *Although some of these definitions are repeated for partial and global maps, it will usually be clear by context whether we are referring to a partial or global map.*

If in addition to preserving relations, f also preserves non-relations, we have two further definitions:

Definition 1.37. If $f : M \to M$ is a monomorphism that preserves non-relations, f is an *embedding* (I').

If f is an endomorphism and an isomorphism, then f is an *automorphism* (A).

Remark 1.38. *The main purpose of defining these maps is that it will allow us to succinctly define the different forms of XY-homogeneity in Chapter 2, as seen in Table 2.1.*

Now we have laid the foundations for our work, we are in a position to think about homogeneity, and how it can apply to structures such as graphs and k-hypergraphs.

Chapter Two

Homogeneity: Fraïssé's Theorem, Graphs and k-Hypergraphs

In this chapter, we will begin to investigate homogeneity. We will state and prove Fraïssé's Theorem, a very powerful tool for investigating homogeneity, before surveying current progress in classifying homogeneous graphs and k-hypergraphs. We will then state a generalisation of homogeneity, known as homomorphism-homogeneity.

2.1 Fraïssé's Theorem

Firstly, I will state and prove Fraïssé's Theorem. The proof I will give here is heavily based on that found in the textbook "Notes on Infinite Permutation Groups" by M. Bhattacharjee, R. Moller, and D. MacPherson [3].

Definition 2.1. Let $\mathcal{L}$ be a language. An *amalgamation class* $\mathcal{C}$ is a non-empty class of finite $\mathcal{L}$-structures with the following properties:

1. Closed under isomorphisms;
2. *(Hereditary Property)* Closed under substructures;
3. *(Joint Embedding Property)* Whenever $A, B \in \mathcal{C}$, there exists D such that $A \leq D$ and $B \leq D$;
4. *(Amalgamation Property)* Whenever $A, B_1, B_2 \in \mathcal{C}$ and there exist embeddings $f_i : A \to B_i$, $i = 1, 2$, then there exists D and embeddings $g_i : B_i \to D$ such that for all $a \in A$ we have $g_1 \circ f_1(a) = g_2 \circ f_2(a)$

Theorem 2.2 (Fraïssé's Theorem)**.** *Let $\mathcal{C}$ be an amalgamation class of finite $\mathcal{L}$-structures. Then:*

a) There exists a homogeneous $\mathcal{L}$-structure (known as the Fraïssé Limit) whose finite substructures are (up to isomorphism) exactly the members of $\mathcal{C}$.

b) Any two homogeneous $\mathcal{L}$-structures as in (a) are isomorphic.

Conversely, if M is a homogeneous $\mathcal{L}$-structure, then the class of finite $\mathcal{L}$-structures which are isomoprhic to substructures of M is an amalgamation class.

Remark 2.3. *For readability, we will split the proof of this into its constituent parts, and prove these as lemmas.*

Lemma 2.4 (Existence). *Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures satisfying conditions* (1)-(4)*. Then there exists a homogeneous $\mathcal{L}$-structure (known as the Fraïssé limit) whose finite substructures are (up to isomorphism) exactly the members of $\mathcal{C}$.*

Proof. The overarching idea of the proof is as follows: we construct our countable structure M by starting with some substructure M_0, and then growing it inductively. We use amalgamation to do this, eventually growing our structure to M, which we can do because the number of amalgamations (applications of the Amalgamation Property) that we need to consider is countable.

Construction: Define $M := \cup_{i \in \mathbb{N}} M_i$, where each M_i is a finite structure and M_{i+1} is "built up from" M_i, in a process we will describe below.

Let K be a countable set of pairs of structures (A, B) such that $A, B \in \mathcal{C}$, and $A \leq B$. Choose K such that it includes all such pairs up to isomorphism, meaning that for each $B \in \mathcal{C}$, K contains each (A, B) where $A \in \mathcal{C}$, and $A \leq B$.

Let $\theta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection such that $i \leq \theta(i, j)$ for all $i, j \in \mathbb{N}$. The purpose of this function is to keep track of which structures we are amalgamating: if $\theta(i, j) = k$, then i will dictate the substructure A already in M, j will dictate the structure B we will amalgamate with A to grow M, and k dictates at which stage this particular amalgamation will occur.

Start with any substructure $M_0 \in \mathcal{C}$.

Then, assuming that M_k has already been defined, list as $(A_{k,j}, B_{k,j}, f_{k,j})_j$ all the triples (A, B, f) where $(A, B) \in K$ and $f : A \to M_k$ is an embedding. This means that A is a substructure of B, and that A is a substructure of M_k. Note that this is why we stipulated that $i \leq \theta(i, j)$ for all $i, j \in \mathbb{N}$: it ensures that when we try to do each amalgamation, the base structure A is already in the structure we are trying to grow, so there is at least one embedding. If there were no amalgamation we could pick, the argument wouldn't work.

We then construct M_{k+1} by applying the amalgamation property. We are amalgamating the identity mapping $id : A_{i,j} \to B_{i,j}$ and the embedding $id_{M_i} \circ f_{i,j} : A_{i,j} \to M_k$ to get M_{k+1}.

This is a countable process that eventually defines M.

Demonstration of homogeneity: Now we have constructed M, we need to show that it satisfies the required properties: that it is homogeneous with finite substructures exactly $\mathcal{C}$.

First, we state and prove property $(**)$, which we will use to show that M has these properties.

$(**)$ - If $A, B \in \mathcal{C}$, $A \leq B$, and $f : A \to M$ is an embedding, then f can be extended to an embedding $g : B \to M$.

Proof. Since $f(A)$ is finite, there exists $j \in \mathbb{N}$ such that $f(A) \leq M_j$. Let the triple (A, B, f) be $(A_{j,l}, B_{j,l}, f_{j,l})$, as listed in K. Define $k := \theta(j, l)$, which means $k \geq j$. Then since M_{k+1} is built by amalgamating M_k and B over $f(A)$, f must extend to some $g : B \to M_{k+1}$. Since M_{k+1} is contained in M, M has property $(**)$. □

From our construction, we can see that the finite substructures of M are exactly the members of $\mathcal{C}$ up to isomorphism. So, we now need to show homogeneity, and we will be done.

Suppose that there exists an isomorphism $f : U \to V$, for $U, V \in \mathcal{C}$. Then, we want to find an automorphism $\hat{f}$ of M extending f. We will do this using what is known as a "back and forth argument", a common technique in the field of homogeneous structures. The idea is to add points one at a time to our partial isomorphism until we can inductively create an automorphism, which we do by adding vertices from the domain and range in turn.

At any step, we have finite substructures U', V' of M and an isomorphism $f' : U' \to V'$ extending f.

At even steps, let $x \in M \setminus U'$. Since M has property $(**)$ and f' is an embedding from U' to M, it can be extended to an embedding f'' defined on $U' \cup \{x\}$. Thus f'' is an isomorphism from $U' \cup \{x\}$ to $V' \cup \{f''(x)\}$. By enumerating the points in M, these steps will ensure that all points in M will be added to the domain of definition, if we choose x to be the next point in the enumeration that isn't currently in U'.

Then at odd steps, we repeat this process, instead adding points to the range of definition. This will ensure that our extension $\hat{f}$ is surjective. With the previous steps, this ensures that $\hat{f}$ is an automorphism, so we are done.

□

Lemma 2.5 (Uniqueness). *Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures satisfying conditions (1)-(4).Then any two homogeneous $\mathcal{L}$-structures as in (a) in Theorem 2.2 are isomorphic.*

Proof. The main idea behind uniqueness is to create an isomorphism ϕ between the two structures M, N by building up a partial isomorphsim to incorporate all points in M and N. We will use what is referred to as a "back and forth" argument to do this, which will ensure our isomorphism contains all points of M and N. We will grow our isomorphic substructures inductively, and then use the fact that M and N are countable to get the complete isomorphism.

Firstly, since M and N are countable, we can write

$$M := \{m_i | i \in \mathbb{N}\}, N := \{n_j | j \in \mathbb{N}\}$$

.

Step 0**:** Define $\phi(m_0) := n_i$, where $i \in \mathbb{N}$ is the least such that $\{m_0\} \cong \{n_i\}$. Let $dom(\phi)$ denote the domain of definition of ϕ at any given step.

Step $2i+1$**:** Suppose that, so far,

$$dom(\phi) = \{m_{k_0}, m_{k_1}, ..., m_{k_{2i}}\}$$

and let $j \in \mathbb{N}$ be the least such that $m_j \notin dom(\phi)$. We will define ϕ on m_j. This will ensure that the process will eventually give us dom(ϕ) $= M$.

Consider $M' = \{m_{k_0}, m_{k_1}, ..., m_{k_{2i}}, m_j\}$.

Since M and N have the same finite substructures, there exists an isomorphic copy of M' in N, such that the first $2i$ points in both are isomorphic too, denoted by $\{n_{l_0}, n_{l_1}, ..., n_{l_{2i+1}}\}$ $= N'$.

However, then

$$\begin{aligned}\{n_{l_0}, n_{l_1}, ..., n_{l_{2i}}\} &\cong \{m_{k_0}, m_{k_1}, ..., m_{k_{2i}}\} \\ &\cong \{\phi(m_{k_0}), \phi(m_{k_1}), ..., \phi(m_{k_{2i}})\}\end{aligned}$$

since ϕ is assumed to be an isomoprhism.

Then, as M is homogeneous, we have that this partial isomorphism is induced by some isomorphism $g \in Aut(N)$. Then, we have that

$$\begin{aligned}\{m_{k_0}, m_{k_1}, ..., m_{k_{2i}}, m_j\} &\cong \{n_{l_0}, n_{l_1}, ..., n_{l_{2i+1}}\} \\ &\cong \{\phi(m_{k_0}), \phi(m_{k_1}), ..., \phi(m_{k_{2i}}), g(n_{l_{2i+1}})\}\end{aligned}$$

with the second isomorphism being due to g.

Hence, we can extend ϕ by defining $\phi(m_j) := g(n_{l_{2i+1}})$, which by the above argument ensures our extension is still an isomorphism.

Step $2i+2$**:** Suppose that, so far,

$$range(\phi) = \{n_{k_0}, n_{k_1}, ..., n_{k_{2i+1}}\}$$

and let $j \in \mathbb{N}$ be the least such that $n_j \notin range(\phi)$. By the same idea as for odd steps, we can extend our isomorphism such that it maps some point to n_j. This will ensure that the process will eventually give us $dom(\phi) = M$.

This process will give us an isomorphism between M and N, completing the proof. □

Remark 2.6. *This result allows us to uniquely identify homogeneous structures: up to isomorphism, there is only one way to form a homogeneous structure out of its finite class of substructures!*

This indicates something very deep about countable homogeneous structures, which is the essential idea used in the proof. There is a natural way to build up the countable structure from its constituent parts if we want homogeneity, and it turns out that this is the only way we can really build it up if we want homogeneity.

Lemma 2.7 (Necessary)**.** *If M is a homogeneous $\mathcal{L}$-structure, then the class of finite $\mathcal{L}$-structures which are isomorphic to substructures of M satisfies* (1) - (4)*.*

Proof. The first property follows immediately: if a finite structure U is contained in $\mathcal{C}$, then any isomorphic copy V must also be isomorphic to a substructure of M, since isomorphism is an equivalence relation, and thus transitive.

The second follows almost immediately: if some finite structure is a substructure of our infinite structure M, then all of its finite substructures must be substructures of M too. So, $\mathcal{C}$ must be closed under substructures.

The third property comes from forming a countable structure from some class of finite ones: if we have two substructures that are contained within our infinite structure M, then there must be some finite structure that contains them both within M, which will be our structure D.

The three properties we have discussed so far are enough to give a countable structure with finite substructures exactly $\mathcal{C}$. The fourth property is where we really use homogeneity.

If A embeds into both B_1 and B_2, then by homogeneity of M, there exists automorphisms of M which extends the inverses of these isomorphisms. Call these ϕ_1, ϕ_2. This essentially means we can "pull back" B_1 and B_2 so that they intersect in A, giving us the amalgamation property. □

With this theorem, we can change our focus to classes of finite substructures that satisfy our four conditions, which are much easier to handle than the countable structures themselves. This also indicates the strength of homogeneity as a property of countable structures, since it gives us "uniqueness" in a sense.

2.2 Homogeneous Graphs and k-Hypergraphs

Now we know how to create homogeneous structures, a natural question arises: what kinds of homogeneous graphs exist?

This would be exceptionally difficult to tackle directly, as it would require knowing both the constituent parts of the countable graph (its finite substructures), as well as how they fit together. However, Fraïssé's Theorem means we can focus just on graphs with suitable constituent parts, meaning we can look only at amalgamation classes.

A natural place to start is by considering the amalgamation class of all finite graphs, which gives us the Rado Graph.

Theorem 2.8. *The class of all finite graphs $\mathcal{C}$ is an amalgamation class. Call the unique homogeneous graph with $\mathcal{C}$ as its age the Rado Graph.*

The following theorem outlines what is known as the *Extension Property*, which can be used to characterise the Rado Graph. Its proof is outlined by Cameron in his survey of the Rado Graph [4], which provides many other interesting and useful properties.

Theorem 2.9. *Consider the following property:*

Given finitely many distinct vertices $u_1, ..., u_m$, $v_1, ..., v_n$, there exists a vertex z which is adjacent to u_1, ..., u_m and nonadjacent to v_1, ..., v_n.

Any graph that satisfies this property is isomorphic to the Rado Graph.

Alongside the Rado Graph, several other homogeneous graphs exist. In fact, they were classified entirely by Lachlan-Woodrow [10] in 1980:

Theorem 2.10. *Let G be a countably infinite graph. Then G is homogeneous if and only if G or its complement is isomorphic to one of the following:*

- *The Rado Graph R,*
- *The graph omitting K^n for $n \geq 3$,*
- *The countable (infinite or finite) disjoint union of infinite complete graphs,*
- *The countably infinite disjoint union of complete finite graphs, all of the same size.*

Now we have looked at graphs, another natural question arises: for a given k, what k-homogeneous graphs exist?

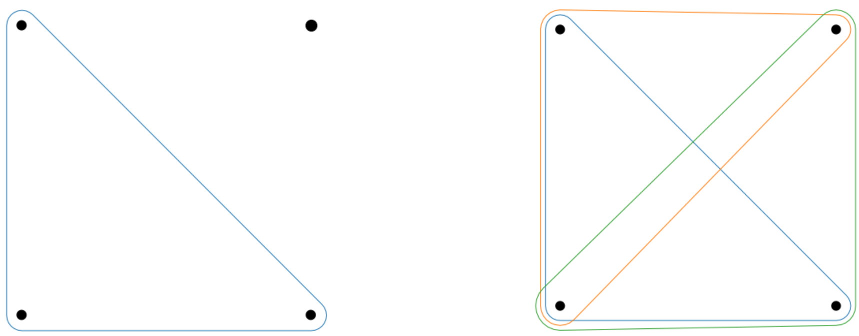
Interestingly compared to the case for $k = 2$, for arbitrary k no simple answer has yet been found, and there are in general many more examples than in the $k = 2$ case. The case with $k = 3$ was explored in more depth by Akhtar and Lachlan in 1995 [1], with some interesting results being as follows:

Theorem 2.11. *There are* $2^{\aleph_0}$ *homogeneous* 3*-hypergraphs.*

Remark 2.12. *This differs from the characterisation of homogeneous graphs in Theorem 2.10, which gave countably many homogeneous graphs.*

Amongst these is the counterpart to the Rado Graph, which is formed by looking at the amalgamation class of all finite 3-hypergraphs (which we will discuss in Chapter 3). One particularly interesting example, due to Akhtar and Lachlan [1]:

Theorem 2.13. *The class of all finite substructures that omit* A_1, A_3 *(see Figure* 2.1*) is an amalgamation class, and so by Fraïssé's Theorem there exists a* 3*-hypergraph with finite substructures exactly those that don't contain* A_1, A_3 *as induced subgraphs.*



(a) A_1 (b) A_3

Figure 2.1: The 3-hypergraphs A_1 and A_3. Notice that omitting these is equivalent to having an even number of edges between any four points.

Remark 2.14. *What makes Theorem 2.13 interesting is that the* 3*-hypergraph it refers to has no counterpart in terms of homogeneous graphs.*

Another interesting result in the $k = 3$ case is again due to Akhtar and Lachlan [1]:

Theorem 2.15. *The* 3*-hypergraph* A_2*, as shown in Figure 2.2, is the unique finite 3-hypergraph of size at least 4 which belongs to every non-trivial infinite amalgamation class of 3-hypergraphs. Specifically, it is contained in every countable homogeneous 3-hypergraph besides the complete and null graphs.*

Here is a related result, which again serves to highlight the difference between the cases $k = 2$ and $k \geq 3$.

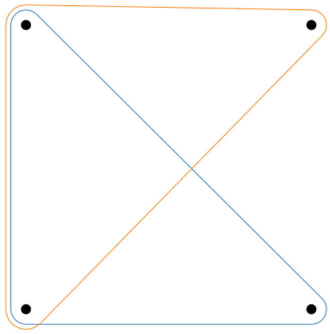


Figure 2.2: The 3-hypergraph A_2.

Theorem 2.16. *Let G be a non-trivial, countable, homogeneous k-hypergraph where $k \geq 3$. Then G has diameter 1. This implies that G is connected.*

Proof. Since G is non-trivial, there exists an edge, say on the vertices $\{x_1, ..., x_k\}$. Then, for any two points u, v in G, we can define an embedding $f : \{x_1, x_2\} \to \{u, v\}$. Since G is homogeneous, f can be extended to an automorphism ϕ. Look at the image of $\{x_1, ..., x_k\}$ under this map. Since ϕ is an automorphism, we have that $\{\phi(x_1), \phi(x_2), ..., \phi(x_k)\} = \{u, v, \phi(x_3), ..., \phi(x_k)\}\}$ is an edge. Thus, u and v are contained in an edge together, and since they were arbitrary vertices in G, we are done. □

Remark 2.17. *Note an interesting difference between diameter in graphs and k-hypergraphs: the only graph with diameter one on a set of vertices V is the complete graph. However, this isn't true for k-hypergraphs.*

Remark 2.18. *If we compare this to the classification of all homogeneous graphs, we can see that this rules out several of its families, including those with a countable disjoint union. This gives an interesting insight: we do not get a simple "chain of inclusion" of homogeneous graphs as we increase the number of points in an edge of our k-hypergraph.*

2.3 Homomorphism-homogeneity

In a similar way to how we generalised graphs earlier, we can also generalise our notion of homogeneity. This allows us to consider a wider array of structures than before. Much of the following is due to the work of Coleman [6].

Firstly, what is homomorphism-homogeneity? Recall our initial definition for homogeneity: any finite partial isomorphism can be extended to an automorphism of the entire relational structure. A stronger version of this condition could arise if we are able to extend all partial maps of a more general variety, and a weaker version could arise if we extend our partial maps to a weaker global map, such as a bimorphism instead of an automorphism. This is the exact idea behind homomorphism-homogeneity: by varying the kinds of partial map we extend, and the kinds of global map we extend to, what new structures can we find?

Definition 2.19. A structure M is *XY-homogeneous* for $X \in \{I, M, H\}$, $Y \in \{H, E, M, B, I', A\}$ if every finite partial map of type X extends to a map of type Y on the whole structure.

Since the first three labelled definitions refer to partial maps, and the final six refer to potential ways to extend each of these kinds of partial maps to global maps, we have eighteen possible ways to combine partial maps with global maps, each giving different kinds of homogeneity. Note how standard homogeneity is IA-homogeneity in this context.

	Global Map Type					
Partial Map Type	H	E	M	B	I'	A
H	HH	HE	HM	HB	HI'	HA
M	MH	ME	MM	MB	MI'	MA
I	IH	IE	IM	IB	II'	IA

Table 2.1: List of all forms of homomorphism-homogeneity.

When we consider homomorphism-homogeneity throughout the rest of this dissertation, we will generally only look at different forms of MY-homogeneity, meaning the initial finite partial map is a monomorphism. One can think of these maps as preserving the number of vertices, and edges, but not necessarily non-edges. Hence, they are intuitively linked to ideas such as subgraphs and sub k-hypergraphs, which we will see later.

2.4 Current State of Research

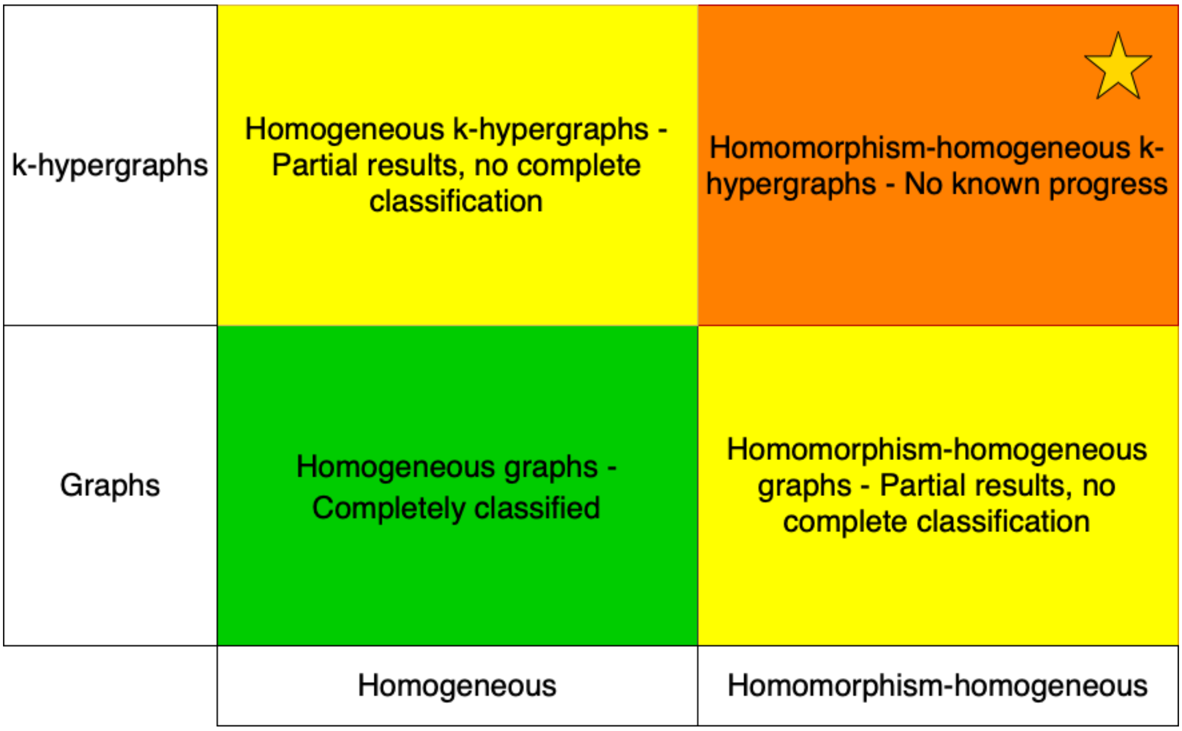


Figure 2.3: Illustration of the current state of research in classifying countable homogeneous and homomorphism-homogeneous graphs and k-hypergraphs.

The current state of the field can be represented as in Figure 2.3. We have covered the left most two sections so far, and we will look at some existing work in the bottom right area later. The rest of this dissertation will be focussed on trying to make progress in the orange section of the diagram, which has previously been neglected.

Chapter Three

Rado k-Hypergraphs

In this chapter, we will generalise the notion of the Rado Graph to the context of k-hypergraphs, and give an extension property that characterises these k-hypergraphs. This extends the characterisation given by Theorem 2.9 for the Rado Graph.

To start, we will give some useful definitions that we will use throughout this chapter.

Definition 3.1. The *age* $\mathcal{C}$ of a k-hypergraph Γ is the class of all finite sub k-hypergraphs of Γ.

Definition 3.2. A k-hypergraph Γ is *universal* if its age $\mathcal{C}$ is the class of all finite k-hypergraphs, for some $k \geq 2$.

Definition 3.3. For any finite $U \subset V(\Gamma)$, let $[U]^{k-1}$ denote the set of all subsets U of size $k-1$.

Then, look at the following condition:

Condition $(*)$**.** *For all* $X, Y \subset [U]^{k-1}$ *such that* $X \cap Y = \emptyset$ *and* $X \cup Y = [U]^{k-1}$*, there exists* $v \in V(\Gamma)$ *such that*

- $\{v\} \cup x \in E(\Gamma)$ *for all* $x \in X$
- $\{v\} \cup y \notin E(\Gamma)$ *for all* $y \in Y$

Remark 3.4. *Notice how when* $k = 2$*, this is equivalent to the characterisation for the Rado Graph given in Theorem 2.9.*

The ultimate aim of this chapter is to use this condition to give a characterisation of universal and homogeneous k-hypergraphs, for each natural number $k \geq 2$. This will allow us to easily ascertain when a k-hypergraph is isomorphic to the universal and homogeneous k-hypergraph, or contains it as a subgraph, which will prove useful in later chapters.

Theorem 3.5. *Consider a k-hypergraph* $\Gamma = (V(\Gamma), E(\Gamma))$*. Then,* Γ *is universal and homogeneous if and only if* Γ *satisfies* $(*)$*.*

We will split the proof of this into three distinct lemmas.

Lemma 3.6. Γ *homogeneous and universal implies that* Γ *satisfies* $(*)$*.*

Proof. Let U, X, Y be as stated in Condition $(*)$, and let Γ_U be the induced k-hypergraph on U. Let U', X', Y' be copies of these respectively. Then let G be a copy of the k-hypergraph we want to form from U, X and Y to satisfy $(*)$:

$$(U' \cup \{v'\}, E(\Gamma_{U'}) \cup \{\{v'\} \cup x | x \in X'\})$$

Then by universality of Γ, there exists an isomorphic copy of G in Γ, which we will denote again by G (we will reuse the labels for U', X', Y' and v').

Then, since Γ_U embeds in G, with embedding we will denote by $\phi|U : U \to G \setminus \{v'\}$, by homogeneity we can extend this embedding to an automorphism of Γ, which we will denote ϕ. Then, define $v = \phi^{-1}(v')$. We claim that this vertex satisfies the required condition of $(*)$.

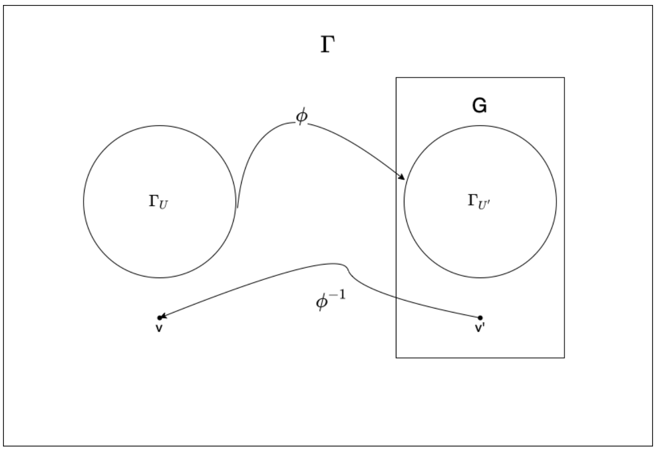


Figure 3.1: Diagram illustrating the proof of Lemma 3.6. The idea is that we embed Γ_U in G, then "pull back" v' using the resulting automorphism to get the desired v.

Let $x \in X$. Then, look at $\phi(x \cup \{v\}) \in G$. From our definition of G this is an edge, since $\phi(x) \in X'$ and $\phi(v) = v'$. Then, since isomorphisms preserves edges, and ϕ is an isomorphism, we have that $x \cup \{v\} = \phi^{-1}(\phi(x \cup \{v\}))$ is an edge, as desired.

The argument for non-edges is almost identical: let $y \in Y$. Then, look at $\phi(y \cup \{v\}) \in G$. From our definition of G this is a non-edge, since $\phi(y) \in Y'$ and $\phi(v) = v'$, and the only edges involving v' in G are with vertices in X, not with those in Y. Then, since isomorphisms preserves non-edges, we have that $y \cup \{v\} = \phi^{-1}(\phi(y \cup \{v\}))$ is a non-edge, as desired.

Hence, X, Y, U and v together satisfy $(*)$. Since X, Y, U were arbitrary, Γ satisfies $(*)$, and we are done.

□

The following proofs are similar to those given by Cameron [4] for the Rado Graph.

Lemma 3.7. Γ *satisfies* $(*)$ *implies that* Γ *is universal.*

Proof. We will prove this by induction.

Let G be a k-hypergraph, with $G = (V, E)$, and enumerate V by $V = \{v_1, v_2, ..., v_{n+1}\}$, so $|V| = n + 1$. We will induct on Γ containing all possible k-hypergraphs as induced k-hypergraphs, for each $n \in \mathbb{N}$.

Base Cases: Assume $n + 1 < k$. Then we can have no edges in the induced k-hypergraph of V with an additional vertex, as edges are relations between k points. Hence, the only k-hypergraph we can have are those of the form $G = (V, \emptyset)$, which is trivially found in the age of Γ by taking any $|V|$ points.

Assume $n + 1 = k$. Then, if G is an edge, we can let $U = X = \{v_1, v_2, ..., v_{k-1}\}$, and $Y = \emptyset$. Then, by the extension property $(*)$ there exists a vertex v such that $X \cup \{v\}$ is an edge, which is thus isomorphic to G, and we are done. The case when G is not an edge follows similarly, by taking $U = Y = \{v_1, v_2, ..., v_{k-1}\}$, and $X = \emptyset$.

Inductive Step: Firstly, we state our inductive hypothesis: assume that every k-hypergraph H with size of vertex set $|V'| = n$ embeds in Γ.

Then, we want to show that any k-hypergraph G with $|V| = n + 1$ embeds in Γ too.

Recall that $V = \{v_1, v_2, ..., v_{n+1}\}$, and let $U = \{v_1, v_2, ..., v_n\}$. Consider the induced k-hypergraph on U, which by the inductive hypothesis embeds in Γ. Denote the points of this embedding by $W = \{v_1', v_2', ..., v_n'\}$, with embedding $f : U \to W$ given by $f(v_i) = v_i'$. Let X be the set of all sets of $k - 1$ points in U that form an edge with v_{n+1} in G, and Y be the set of all sets of $k - 1$ points in U that form a non-edge with v_{n+1} in G.

Let X', Y' correspond to X and Y as subsets of W (replacing each v_i with v_i'). Then, we have that W, X', Y' satisfy the requirements for $(*)$, so we have that there exists a vertex we will denote by v' in Γ as specified by $(*)$.

Now, we claim that the induced graph on $W \cup \{v'\}$ is isomorphic to G. Let $\phi : V \to W \cup \{v'\}$ be given by f on vertices in U, and define $\phi(v_{n+1}) = v'$. Now, any edge or non-edge in G not including v_{n+1} is preserved by ϕ, since f is an isomorphism. Any edge in G including v_{n+1}, denoted by A, is also preserved, since it will contain exactly $k - 1$ points in U. Thus, $A \in X$, and hence $f(A) \in X'$, and will therefore form an edge with v'. The same argument works for non-edges, by considering $B \in Y$. Hence, ϕ is an isomorphism (it is trivially a bijection), so we are done.

This proves the inductive step, and hence universality follows by induction.

□

Lemma 3.8. Γ *satisfies* $(*)$ *implies that* Γ *is homogeneous.*

Proof. We will make use of a "back and forth argument" explicitly, very similarly to Lemma 2.5 (notice that we used a "forth argument" in proving universality, that was hidden in the induction).

First, since Γ is a countable homogeneous structure, let us enumerate the vertices V of Γ as $\{v_1, v_2, v_3, ...\}$.

Let $\phi : A \to B$ be a finite partial isomorphism within Γ.

We will specify a process with countably many steps, which will build up ϕ into an automorphism. To do this, we will need to ensure that after each step our expanded function is still a partial isomorphism, and that every point of V is eventually contained in both its range and domain, to ensure that our resultant function is a bijection from Γ to itself. This will show that our resulting function is an automorphism.

Step $(2m)$: Suppose that, so far,

$$dom(\phi) = \{v_{k_0}, v_{k_1}, ..., v_{k_i}\}$$

and let $m \in \mathbb{N}$ be the least such that $v_m \notin dom(\phi)$. We will define ϕ on v_m. This will ensure that the process will eventually give us $dom(\phi) = V$.

Now, let X be the neighbourhood of v_m in A, and let Y be the neighbourhood of v_m in $\bar{A}$ (giving us all non-edges in A). Then, define $X' := \phi(X)$, $Y' := \phi(Y)$. By applying Condition $(*)$ to X' and Y', there must exist some vertex in Γ that forms the necessary edges and non-edges with X' and Y'. Denote this vertex by v'_m.

Then, if we redefine A to include v_m and B to include v'_m, and expand the domain of the partial map $\phi : A \to B$ such that $\phi(v_m) = v'_m$, ϕ is still an embedding. This follows by the same justification as in the inductive step in Lemma 3.7.

Step $(2m + 1)$: Suppose that, so far,

$$range(\phi) = \{v_{l_0}, v_{l_1}, ..., v_{l_i}\}$$

and let $j \in \mathbb{N}$ be the least such that $v_j \notin range(\phi)$. Using a similar argument to that for the even steps gives us a point v'_j such that we can define $\phi(v'_j) = v_j$ whilst still being an isomorphism. This will ensure that the process will eventually give us $range(\phi) = V$.

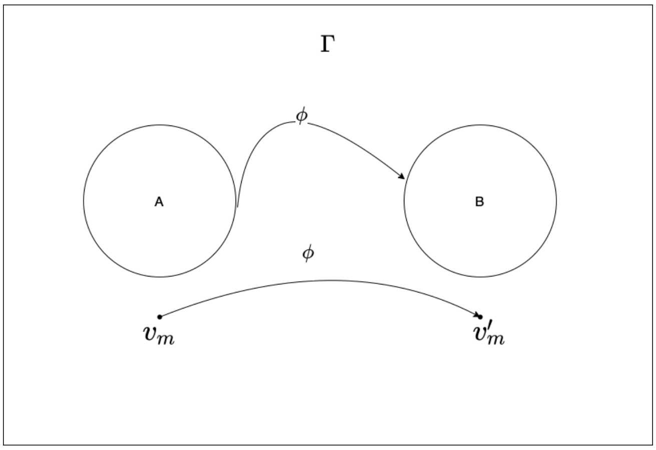


Figure 3.2: Diagram illustrating the even steps of the proof of Lemma 3.8. The idea is that we can use $(*)$ to find a point v'_m that acts the same with respect to B as v_m does with respect to A, and then extend ϕ by mapping v_m to v'_m.

Hence, we can extend ϕ to an automorphism of Γ. Since ϕ was an arbitrary embedding, the structure is homogeneous, and we are done.

□

Note that for any two k-hypergraphs that are homogeneous and have the same age, we can apply Fraïssé's Theorem to show that they are isomorphic. Then by Theorem 3.5, any k-hypergraphs that satisfy $(*)$ are also isomorphic. We will refer to any k-hypergraph satisfying $(*)$ from now on as the Rado k-hypergraph, since universality and homogeneity are the defining properties of the Rado Graph.

Definition 3.9. The *Rado k-hypergraph*, referred to as R_k, is the unique (up to isomorphism) k-hypergraph satisfying $(*)$.

Chapter Four

Properties of Homomorphism Homogeneous k-Hypergraphs

The overarching question we are trying to answer here is the following: as we generalise graphs to k-hypergraphs, which properties of homomorphism-homogeneous graphs remain the same, and what new structure do we introduce? Here, we will initially focus on MY-homogeneity for $Y \in \{H, M, B\}$, before narrowing our focus to MB in the next chapter.

4.1 Basic Properties of MY-homogeneous k-hypergraphs

In their 2006 paper [5] (Section 2), Cameron and Nešetřil provided several basic theorems for MY-homogeneous graphs. In this section, we will investigate some analogues for several of the theorems proved there, in the context of k-hypergraphs.

We start with a simple characterisation of finite MM-homogeneous, and hence also MB-homogeneous, k-hypergraphs:

Lemma 4.1. *The only finite MM-homogeneous k-hypergraphs are the null and complete k-hypergraphs, for each k.*

Proof. Suppose for a contradiction that G is a finite k-hypergraph that isn't null or complete, with n edges. Then, there exists an edge, and a non-edge. The map from some non-edge to some edge is a monomorphism. If G were MM, we could extend this to a monomorphism of all of G. However, as we extend our map by adding more vertices, we will eventually have to map an edge to a non-edge, as we need to now map the remaining n edges to $n-1$ edges injectively, which isn't possible. This means our eventual map won't be a monomorphism, and thus G isn't MM-homogeneous.

We can see that the null and complete k-hypergraphs are MM-homogeneous, since we can extend any partial monomorphism by simply mapping the unused points in the domain to the unused points in the range in any way, completing the proof.

□

The following properties are a generalisation of results by Cameron and Nešetřil [5] for MM-homogeneous graphs. However, their proof wasn't able to generalise to this result, forcing a different approach which resembles the proof of Lemma 4.1.

Lemma 4.2. *Every non-trivial, countably infinite MM-homogeneous k-hypergraph G contains a complete k-hypergraph of size n for each $n \geq k$.*

Proof. Let $n \geq k$, and let A be any finite induced k-hypergraph of G with n vertices. If A is complete, we are done. Otherwise, there exists a non-edge in A, and some edge in G, since G is non-trivial. Then, map the non-edge to this edge, and denote this map f. Since G is MM-homogeneous, we can extend f to a global monomorphism, call it $\phi : G \to G$.

Then, look at $\phi(A)$. This must have strictly more edges than A, since it is a homomorphism and thus preserves edges, and a non-edge has been mapped to an edge. Since ϕ is injective, $\phi(A)$ has the same number of vertices as A. Repeat this process, replacing A with $\phi(A)$, whilst there are still non-edges in $\phi(A)$. There are only finitely many non-edges in A, so after finitely many iterations this gives us a complete k-hypergraph on the same number of vertices as A, n. Since n was arbitrary, we are done.

□

Theorem 4.3. *Every non-trivial, countably infinite MM-homogeneous k-hypergraph contains an infinite complete k-hypergraph.*

Proof. Let $a_1, ..., a_k$ be the vertices of an edge in G, and denote this edge by A_k. Then, the idea is that we can inductively grow our infinite complete k-hypergraph from these vertices.

By Lemma 4.2, there is a complete k-hypergraph of size n for each $n \geq k$. Let A_{k+1} be the complete k-hypergraph on $k+1$ points. Let f refer to the mapping of k points of A_{k+1} to A_k, and label the unmapped point of A_{k+1} by u. Since f is an injective homomorphism, we can extend it to a global, injective homomorphism, say ϕ.

Then, look at $\phi(u)$: this must form a complete k-hypergraph on $k+1$ points with $\phi(A_k)$. Label the point $\phi(u)$ with a_{k+1}. This now gives us a complete k-hypergraph on $k+1$ vertices, using the starting points $a_1, ..., a_k$.

Repeat this process of mapping larger and larger complete k-hypergraphs onto an existing complete k-hypergraph countably many times. This will produce a countably infinite k-hypergraph, as desired. □

However, many of the other examples given by Cameron and Nešetřil don't work for k-hypergraphs. This is due to the following result for k-hypergraphs, which mirrors Theorem 2.16 for the case of standard homogeneity.

Theorem 4.4. *Every non-trivial countable MH-homogeneous k-hypergraph G has diameter 1. In particular, all MH-homogeneous k-hypergraphs are connected for $k \geq 3$.*

Proof. This proof proceeds almost identically to that of Theorem 2.16.

Since G is non-trivial, there exists an edge, say on the vertices $\{x_1, ..., x_k\}$. Then, for any two points u, v in G, we can define a monomorphism $f : \{x_1, x_2\} \to \{u, v\}$. Then, since G is MH-homogeneous, we have that f can be extended to a global homomorphism ϕ. Now look at the image of $\{x_1, ..., x_k\}$ under this map. Since ϕ is a homomorphism, we have that $\{\phi(x_1), ..., \phi(x_k)\}$ is an edge. Thus, since u, $v \in \{\phi(x_1), ..., \phi(x_k)\}$, they are in an edge together. Since they were arbitrary vertices in G, we are done. □

The key reason this extends to MH-homogeneity is that homogeneity guarantees that an edge must be mapped injectivity to another edge, meaning that k points forming an edge must be mapped to k distinct points.

4.2 R_k as a spanning sub k-hypergraph

Next, we will give some results relating MB-homogeneity to containing the Rado k-hypergraph as a spanning sub k-hypergraph. This will begin by looking at the condition we used to characterise Rado k-hypergraphs in Chapter 3, and making it relevant to sub k-hypergraphs. Many of these proofs are simple generalisations of previous work by Cameron and Nešetřil [5].

The following is a property that will allow us to characterise when R_k spans a k-hypergraph $\Gamma = (V, E)$.

Condition (†). *For any collection of points $v_1, ..., v_n$ in V, there exists a point u such that every collection of $k-1$ points of v_i forms an edge with u.*

Note that Γ satisfies $(*)$, the Extension Property from Chapter 3, implies that Γ satisfies (†). This makes sense in relation to Theorem 3.5, since trivially the Rado k-hypergraph is a spanning sub k-hypergraph of itself.

Lemma 4.5. *A countable k-hypergraph Γ contains R_k as a spanning sub k-hypergraph if and only if Γ satisfies* (†).

Proof. The forward direction follows from Theorem 3.5. Assume that Γ contains R_k as a spanning sub k-hypergraph. Then, Theorem 3.5 implies that $(*)$ holds for R_k. If we take

$X = U$ as in Theorem 3.5, then we have that (†) holds in R_k. Hence, (†) holds in Γ, since introducing edges cannot make the property false.

For the reverse direction, let Γ be a k-hypergraph satisfying (†). We will construct a bimorphism from R_k to Γ, which is equivalent to showing R_k is a spanning sub-hypergraph of Γ. This will use another back and forth argument, as seen several times already.

To start, enumerate the vertices of R_k as $v_1, v_2, ...$, and the vertices of Γ as $u_1, u_2, ...$. Then, map $v_1, ..., v_{k-1}$ to $u_1, ..., u_{k-1}$, and call this map f. We will expand the map f recursively. Denote the current domain of f as V, the current range of f as U.

At odd stages, extend f by taking the next unmapped point in R_k, say v_n, and mapping it to a point u that satisfies the condition (†) in Γ with respect to $u_1, ..., u_{k-1}$, which exists by assumption. To see that the map after this point is still a monomorphism, consider all new possible edges in the domain of f, which are the edges involving v_n. These all map to edges by the use of (†), so f still maps all edges to edges, and hence is a monomorphism.

At even stages, extend f by looking at the next point in the enumeration of Γ not yet in the range of f, say u_n. Then, we use the full characterisation of R_k, $(*)$, to pick a point v'_n that behaves with V exactly like u_n does with respect to U. Then, extend f by mapping v'_n to u_n, which is still a monomorphism by choice of v'_n. Note that we have to use the full property here because use of the weaker (†) could give us an edge that maps to a non-edge, so it wouldn't be a homomorphism.

Doing countably many steps of the odd and even stages will ensure that the domain and range of f is all of R_k and Γ, respectively. Thus, we can construct the required bimorphism, concluding the proof.

□

Lemma 4.6. *Any k-hypergraph containing R_k as a spanning subhypergraph is HH and MM, and thus MH.*

Proof. If R_k is a spanning sub k-hypergraph of a k-hypergraph Γ, then the property (†) holds. Let f be a partial homomorphism or monomorphism of R_k, with domain U and range V. Then, we can build up f to make it global by just using the same idea as the odd steps (or the "forth argument") from the proof of Lemma 4.5. This means that for each new vertex u we introduce in the domain, there exists a new vertex v in the range that is in every possible edge with V by (†). We map u to v, and we still have a homomorphism. Doing this repeatedly gives us a global homomorphism or monomorphism, so we are done.

□

Remark 4.7. *This isn't true for MB-homogeneous k-hypergraphs in general, as shown in the remarks of [6] after Prop 8.1.6 (the given example is the complement of the disjoint* *union of countably many finite complete graphs). The problem here is that we need the "back" direction to extend the map to a bimorphism, but* (†) *is insufficient to provide this, as noted in the proof of Lemma 4.5.*

4.3 Intersection Between IA and MB Homogeneous k-Hypergraphs

Due to the incomplete classification of the homogeneous k-hypergraphs for $k \geq 3$, the classification of both MB and IA k-hypergraphs, as provided by Coleman [6] for graphs, seems out of reach. However, the disanalogy alluded to in Remark 2.18 extends here to a disanalogy to the classification of MB and IA homogeneous k-hypergraphs: several of the examples given by [6] are disconnected, so have no counterpart in k-hypergraphs.

This hints at a recurring theme we have seen during this chapter: although the k-hypergraph form of many results for homomorphism-homogeneous graphs do hold, there are some qualitatively different results here, and the proofs do at times require genuinely different approaches.

Chapter Five

Examples of Homomorphism Homogeneous k-Hypergraphs

In this section, in a similar vein to Coleman [6], we will first give some sufficient conditions for MB-homogeneous k-hypergraphs, before giving some examples. We will use this to generate first countably many distinct MB-homogeneous k-hypergraphs for each k, and then uncountably many for each k.

We will also define an equivalence relation, which we will use to help characterise MB-homogeneous k-hypergraphs:

Definition 5.1. The k-hypergraphs Γ_1 and Γ_2 are *bimorphism equivalent* if there exist bijective homomorphisms (bimorphisms) $\alpha : \Gamma_1 \to \Gamma_2$ and $\beta : \Gamma_2 \to \Gamma_1$

Remark 5.2. *The definition of homomorphism here means that a bijective homomorphism isn't necessarily an isomorphism, unlike in many other contexts in mathematics, so bimorphisms are distinct from isomorphisms. This is because bimorphisms must preserve edges, but not necessarily non-edges.*

5.1 Sufficient Conditions

Firstly, we give sufficient conditions for a graph to be MB-homogeneous, which are simple generalisations of conditions used in Coleman's work [6]. Note that the first condition is exactly (†), which gave us R_k as a spanning sub k-hypergraph in Theorem 4.5. Introducing the second condition will allow us to do the back direction we noticed an issue with in Remark 4.7. Hence, we can think of these conditions as giving us k-hypergraphs with R_k as a spanning sub k-hypergraph, with the extra structure needed to give us MB-homogeneity.

Condition (Δ)**.** *For any collection of points $v_1, ..., v_n$, we can find a point v such that every collection of k-1 points of v_i forms an edge with v.*

Condition $(\therefore)$**.** *For any collection of points $v_1, ..., v_n$, we can find a point u such that every collection of $k-1$ points of v_i forms a non-edge with u.*

Theorem 5.3. *If a k-hypergraph Γ satisfies (Δ) and $(\therefore)$, then it is MB-homogeneous.*

We will only sketch the proof, since it follows a back and forth argument we have seen several times.

Proof. Let $f : U \to V$ be a partial monomorphism, and enumerate $\Gamma(V)$ by $v_1, v_2, \ldots$. We will extend f to a global bimorphism through a process with countably many steps, completing the proof.

On odd steps, take the next point in the enumeration $v_1, v_2, ...$ not yet used in the domain of the map, say u. Then, by (Δ) we can find a point that is "joined to every possible set of $k-1$ points" in V, say v. Then, extending f by mapping u to v guarantees we still have a monomorphism (every new set of k points is mapped to an edge, so all possible edges are preserved, and so we are still a homomorphism).

On even steps, take the next point in the enumeration $v_1, v_2, ...$ not yet used in the range of the map, say v. Then, we can use $(\therefore)$ to find a point that "forms a non-edge with every possible set of $k-1$ points" in U, say u. Then, extending f by mapping u to v still gives us a monomorphism (every new potential edge in the range is mapped to by a non-edge, so we are still a homomorphism).

This construction will extend f to a bijective homomorphism, since the odd steps ensure we have all of $\Gamma(V)$ in the domain, and the even steps ensure we have all of $\Gamma(V)$ in the range, and thus we are done. □

Next, we will give two results that will help us to find further MB-homogeneous hypergraphs using these conditions. The proofs are again similar to the one given above, and those given by Coleman [6], so we will be brief.

Theorem 5.4. *Let Γ_1, Γ_2 be bimorphism equivalent (with bimorphisms $\alpha : \Gamma_1 \to \Gamma_2$ and $\beta : \Gamma_2 \to \Gamma_1$). Then Γ_1 satisfies (Δ) and $(\therefore)$ if and only if Γ_2 satisfies (Δ) and $(\therefore)$.*

Proof. Without loss of generality, if we suppose that Γ_1, Γ_2 are bimorphism equivalent and that Γ_1 satisfies (Δ) and $(\therefore)$, then it is sufficient to show that Γ_2 satisfies (Δ) and $(\therefore)$.

Firstly, we will show that Γ_2 satisfies (Δ). To start, let Y be any finite set of points in Γ_2. Since α is a bijection, there exists $X \subset V(\Gamma_1)$ such that $\alpha(X) = Y$. Then, by assumption there exists some vertex u that satisfies (Δ) for X. Then, since α is a homomorphism, $\alpha(u)$ must satisfy (Δ) for Y. Y was arbitrary, so we have that Γ_2 satisfies (Δ).

To show that Γ_2 satisfies $(\therefore)$, we apply the same argument, but instead we look at the inverse of β. This must preserve non-edges, and thus we can use $(\therefore)$ in Γ_1 to give us $(\therefore)$ in Γ_2. □

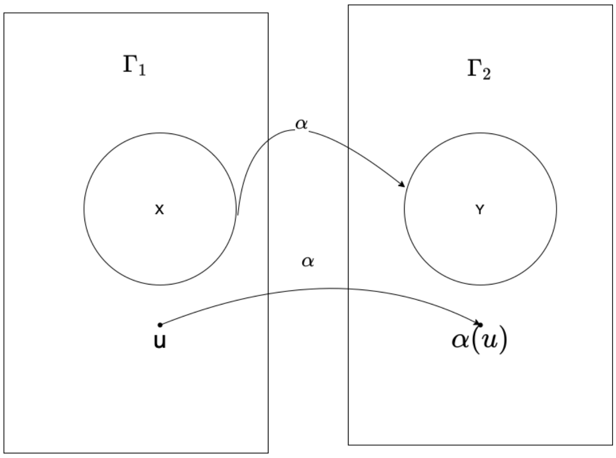


Figure 5.1: Illustration of the proof of Theorem 5.4, demonstrating why Γ_2 satisfies (Δ). The idea is that if u satisfies (Δ) with respect to X, since α is a bimorphism, $\alpha(u)$ must satisfy (Δ) with respect to Y.

Theorem 5.5. *If Γ_1, Γ_2 both satisfy (Δ) and $(\therefore)$, then Γ_1, Γ_2 are bimorphism equivalent.*

Proof. We will outline a back and forth argument that is very similar to that in Theorem 5.3. We will only give a bimorphism from Γ_1 to Γ_2. The other bimorphism exists by a symmetric argument.

Start by mapping any $(k-1)$ points of Γ_1 to any $(k-1)$ points of Γ_2, and call this initial map f. Enumerate $\Gamma_1(V)$ by $v_1, v_2, ...$, and enumerate $\Gamma_2(V)$ by $u_1, u_2, ...$.

At each even step, take the next point in the enumeration $v_1, v_2, ...$ not yet used in the domain of the map f, say v. Then, use (Δ) in Γ_2 to find a suitable point to map v to.

At each odd step, take the next point in the enumeration $u_1, u_2, ...$ not yet used in the range of the map f, say u. Then, use $(\therefore)$ in Γ_1 to find a suitable point to map to u.

Each step preserves being a homomorphism, and doing countably many even and odd steps ensures we eventually have a bijection, so we are done. □

Remark 5.6. *Just as Coleman did, we can note that since R_k satisfies (Δ) and $(\therefore)$, we now have access to a whole range of MB-homogeneous structures: all of those bimorphism equivalent to R_k.*

5.2 Countably Many Examples

Next, we will use these sufficient conditions to create countably infinitely many MB-homogeneous k-hypergraphs. We will do this by extending the construction used by Coleman [6] to create MB-homogeneous k-hypergraphs from a binary sequence.

Definition 5.7. A sequence $P = (p_i)_{i\in\mathbb{N}}$ is *binary* if each $p_i \in \{0, 1\}$ for all $i \in \mathbb{N}$.

Example 5.8. Let $P = (p_i)_{i\in\mathbb{N}}$ be a binary sequence, with the first $k-1$ entries 1s.

Define the k-hypergraph $\Gamma(P)$ on the vertex set $\{v_0, v_1, ...\}$ such that $\{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$ is an edge if and only if $p_{max(i_1,i_2,...,i_{i_k})} = 0$.

From this, we can observe that:

- If $p_i = 0$, then $\{v_{i_1}, v_{i_2}, ..., v_{i_{k-1}}, v_i\}$ is an edge for any distinct $i_1, i_2, ..., i_{k-1} < i$
- If $p_i = 1$, then $\{v_{i_1}, v_{i_2}, ..., v_{i_{k-1}}, v_i\}$ is not an edge for any distinct $i_1, i_2, ..., i_{k-1} < i$

Figure 5.2 is an image from Coleman that shows the graph version of this construction. The k-hypergraph version is harder to visualise, so we will use images for graphs instead, but most of the intuition carries over.

We will also introduce some notation, again due to Coleman: the first section of consecutive 1s is VI_1, the first consecutive 0s is VO_1, then VI_2, $V0_2$, VI_2, and so on.

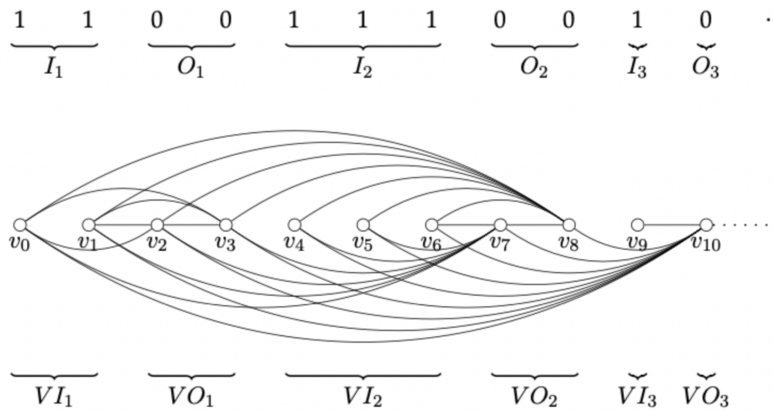


Figure 5.2: Illustration of the construction of a graph from a binary sequence, as outlined in Example 5.8, along with the definition of VI's, and VO's. The general k-hypergraph construction is very similar. Image from Coleman [6].

Lemma 5.9. *If a binary sequence $P = (p_i)_{i\in\mathbb{N}}$ has infinitely many* 0*s and* 1*s then* $\Gamma(P)$ *satisfies* (Δ) *and* ($\therefore$).

Proof. Let $v_{i_1}, ..., v_{i_n}$ be a collection of points in $\Gamma(P)$. Then, since we have infinitely many 0s, there exists $j > i_1, i_2, ..., i_{k-1}$ such that $p_j = 0$. Then, as we observed, p_j will form an edge with any collection of $k-1$ points from $v_{i_1}, ..., v_{i_n}$. This choice of n points was arbitrary, so $\Gamma(P)$ satisfies (Δ).

To show ($\therefore$), repeat the above argument by choosing some $j > i_1, i_2, ..., i_{k-1}$ such that $p_j = 1$. ☐

Now we have a method of constructing MB-homogeneous k-hypergraphs from binary sequences, we will give a countable family of binary sequences that give rise to non-isomorphic k-hypergraphs.

Example 5.10. Consider the sequence $P_n = (p_i)_{i\in\mathbb{N}}$, for $n \geq k$, given by

$$p_i = \begin{cases} 0 & \text{i} = n, n+2, n+4, ... \\ 1 & \text{i} = 0, 1, ..., k-1, k, ..., n-1, n+1, n+3, ... \end{cases}$$

We will show that these sequences give us countably many non-isomorphic, but bimorphism equivalent MB-homogenous k-hypergraphs.

Theorem 5.11. *There exist countably many non-isomorphic, bimorphism equivalent MB-homogenous k-hypergraphs.*

Proof. Firstly, consider the formula given below:

$$\phi_k(x) = (\exists y \in V(\Gamma_k(P_n)))(\neg(y = x))(\forall\{z_1, ..., z_{k-1}\} \subset V(\Gamma_k(P_n)) \setminus \{y, x\})$$
$$(\{z_1, ..., z_{k-1}, x\} \in E(\Gamma_k(P_n)) \iff \{z_1, ..., z_{k-1}, y\} \in E(\Gamma_k(P_n)))$$

This is equivalent to asking, for a given vertex x, "is there another distinct vertex y that is involved in the same edges as x". Note a slight complication here that doesn't arise in the graph case: we can determine perfectly the edges of a vertex in the graph case by looking at its neighbours, but the same isn't true for k-hypergraphs. This is why we only consider edges not containing x and y, but the proof will still work.

Now, we claim that for each P_n, the vertices that satisfy this are exactly VI_1.

Lemma 5.12. *$x \in VI_1$ implies $\phi_k(x)$ is true.*

Proof. Let $x = v_i \in VI_1$. Then, consider $x \neq v_j = y \in V$, defined by

$$j = \begin{cases} 1 & \text{i} \neq 1 \\ k-1 & \text{i} = 1 \end{cases}$$

Now, suppose that $\{v_{i_1}, v_{i_2}, ..., v_{i_{k-1}}\} \subset V(\Gamma_k(P_n)) \setminus \{y, x\}$ and $\{v_{i_1}, v_{i_2}, ..., v_{i_{k-1}}, x\} \in E(\Gamma_k(P_n))$. By our definition of P_n, this implies $p_{max(i_1,...i_{k-1},i)} = 0$. But since $p_i = 1$ (as v_i corresponds to x), this must mean that $p_{max(i_1,...i_{k-1})} = 0$.

Then, since $y = v_j$, and $j < max(i_1, ...i_{k-1})$ by definition ($y \in VI_1$, and one of these points must not be in VI_1 since $p_{max(i_1,...i_{k-1})} = 0$), we have that $p_{max(i_1,...i_{k-1},y)} = p_{max(i_1,...i_{k-1})} = 0$, and hence y forms an edge with these $k-1$ points. Since $v_{i_1}, v_{i_2}, ..., v_{i_{k-1}}$ were arbitrary, we are done. □

Now, we will show that these are the only points that satisfy this condition.

Lemma 5.13. *$x \notin VI_1$ implies $\phi_k(x)$ is false.*

Proof. **Case 1:** Assume $x = v_i$ such that $p_i = 0$.

Then, for any point $y = v_j$ such that $p_j = 1$, the formula won't be satisfied: any $k-1$ points before x and y will form an edge with x, but not with y. If there are not $k-1$ points before y, then we must have that $j \leq k-1$. Then, consider the first k points, without y: this gives $k-1$ points that form a non-edge with y, but an edge with x, so again we are done.

For any point $y = v_j$ such that $p_j = 0$, there exists some vertex $u = v_l$ and $p_l = 1$ such that either $i < l < j$ or $j < l < i$ ("between x and y"). Then, take any $k-2$ points that are all before x and y, and u. If (without loss of generality) y is "after" x, we have that these $k-1$ points with y form an edge, but the $k-1$ points with x do not, since u is after x and $p_u = 1$.

Hence, for all choices of y, the formula doesn't hold when $p_x = 0$.

Case 2: Assume $x = v_i$ such that $p_i = 1$, and $x \notin VI_1$.

For any point $y = v_j$ such that $p_j = 0$, taking any $k-1$ points before x and y will form an edge with y, but not x, so the formula isn't satisfied.

For any point y such that $p_j = 1$, take any point that is 0 "between" points x and y, along with $k-2$ other points before x and y. This will give an edge when incorporated with x, but not with y, so once again the formula doesn't hold. Note we can pick these points because x isn't in VI_1. □

Thus, exactly n points in the k-hypergraph generated by P_n satisfy ϕ_k because $|VI_1| = n$, and hence $n \neq m$ implies that P_n and P_m are not isomorphic. Since all the k-hypergraphs generated from the sequences P_n satisfy (Δ) and $(\therefore)$, and hence are bimorphism equivalent, the result follows.

□

5.3 Uncountably Many Examples

Finally, we will extend Theorem 5.11 to find uncountably many examples, by extending another of Coleman's [6] constructions.

Definition 5.14. Let $A = (a_i)_{i \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers, with the additional condition that $a_1 \geq k+2$. Then, define PA to be the binary sequence formed by taking a_0 1s followed by a single 0, then a_1 1s, then a 0, and so on. Then, using the construction in Example 5.8, we have a corresponding k-hypergraph, which we will denote by $\Gamma(PA)$.

The idea is to introduce (k, n)-cycles, as defined in Definition 1.18, into these k-hypergraphs $\Gamma(PA)$. We will add them in such a way that each sequence gives us a k-

hypergraph that contains different finite induced sub k-hypergraphs, giving us uncountably many non-isomorphic k-hypergraphs, for each k.

The following definition outlines how to add these (k, n)-cycles.

Definition 5.15. Given $\Gamma(PA)$ as defined previously, define $\Gamma(PA)'$ to be $(V(\Gamma(PA)), E(\Gamma(PA)) \cup E)$, where E consists of a (k, n)-cycle between each consecutive finite subsequence of 1s. Intuitively, for each consecutive set of independent points in $\Gamma(PA)$, add a (k, n)-cycle.

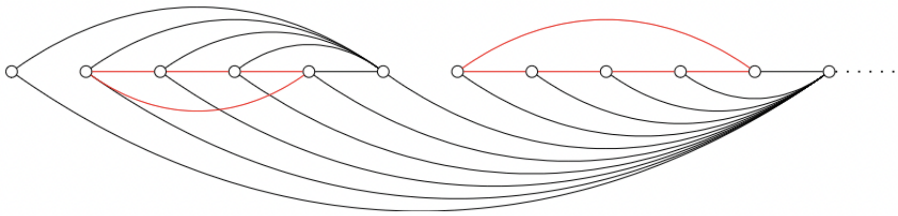


Figure 5.3: Illustration of going from the original graph $\Gamma(PA)$, to the graph with added cycles $\Gamma(PA)'$, as outlined in Definition 5.15. The additional cycles are highlighted here in red. Image from Coleman [7].

When we add these cycles, we need to check that the resulting k-hypergraphs are still MB-homogeneous. The following Lemma guarantees this.

Lemma 5.16. *$\Gamma(PA)'$ satisfies (Δ) and $(\therefore)$.*

Proof. Since $\Gamma(PA)$ is a sub k-hypergraph of $\Gamma(PA)'$, and $\Gamma(PA)$ satisfies (Δ), $\Gamma(PA)'$ also satisfies (Δ).

To see that $(\therefore)$ still holds, let $v_{i_1}, ..., v_{i_n}$ be a collection of points in $\Gamma(P)$. Repeat the same argument as in the proof of Lemma 5.9, but take the v_j to be in a different section of consecutive points past all the $v_{i_1}, ..., v_{i_n}$. Then, when going from $\Gamma(PA)$ to $\Gamma(PA)'$, we haven't introduced any edges between the v_j and the previous points, so $(\therefore)$ still holds. $\square$

Next, we will prove a handful of Lemmas that we will need later:

Lemma 5.17. *Every vertex of a (k, n)-cycle has degree k.*

Proof. Let C_n^k be a (k, n)-cycle, defined on vertices $V = \{x_0, x_1, ..., x_{n-1}\}$ as in Definition 1.18. Consider x_i, where $0 \leq i \leq n - 1$. Then, we have that

$$\{x_i, x_{(i+1)\text{mod n}}, ..., x_{(i+k-1)\text{mod n}}\}, \{x_{(i-1)\text{mod n}}, x_i, ..., x_{(i+k-2)\text{mod n}}\},$$
$$..., \{x_{(i-k+1)\text{mod n}}, ..., x_{(i-1)\text{mod n}}, x_i\}$$

are all the distinct edges containing x_i. There are k of these, giving us the result. $\square$

Lemma 5.18. *There are no (k,n)-cycles for $n \geq k+2$ as induced k-hypergraphs of $\Gamma(PA)$.*

Proof. Suppose, for a contradiction, that X is an induced k-hypergraph on some vertices $\{v_{i_1}, v_{i_2}, ..., v_{i_n}\}$ for $n \geq k+2$, with associated subsequence P' given by $i_1 < i_2 < ... < i_n$, that forms a (k,n)-cycle. Since each vertex in a (k,n)-cycle has degree k, we must have that $p_{i_j} = 1$ for each $i_j > i_{k+1}$. But then we must have that $p_{i_{k+2}} = ... = p_{i_n} = 1$, so $v_{i_{k+2}}$ has degree 0. This is a contradiction, and so X is not a (k,n)-cycle with $n \geq k+2$.

□

Lemma 5.19. *If C_m^k, C_n^k are (k,m) and (k,n)-cycles of length m and n respectively, then one embeds in the other if and only if $m = n$.*

Proof. First, assume without loss of generality that $m \leq n$, and that C_m^k embeds in C_n^k. Denote the vertices of C_n^k by $v_0, v_1, ...v_{n-1}$ such that any consecutive k points form an edge.

Let e be an edge of C_m^k. This must also be an edge of C_n^k, so it must be between k consecutive points, say $v_i, ..., v_{(i+k-1) \bmod k}$.

There are exactly two edges in C_m^k that intersect with e across exactly $k-1$ points. There are also exactly two edges in C_n^k that intersect with e across exactly $k-1$ points, namely $\{v_{(i-1) \bmod k}, ..., v_{(i+k-2) \bmod k}\}$ and $\{v_{(i+1) \bmod k}, ..., v_{(i+k) \bmod k}\}$. Hence, these must be the same in C_m^k.

Proceeding with this "around the cycle C_n^k" eventually gives that all of the edges in C_n^k are in C_m^k. Since we assumed that $m \leq n$, we must have that $C_m^k = C_n^k$, and thus we are done.

□

Now, the following proposition essentially states that we don't accidentally introduce any unexpected (k,n)-cycles - with this, our main result will follow easily.

Lemma 5.20. *Suppose that $A = (a_i)_{i \in \mathbb{N}}$, with $a_1 \geq k+2$, and that $m \geq k+2$ is an integer such that $m \neq a_i$ for all i. Then, $\Gamma(PA)'$ doesn't contain a (k,m)-cycle.*

Proof. Suppose that M is a (k,m)-cycle and a sub k-hypergraph of $\Gamma(PA)'$, induced by the finite subsequence $(q_{i_1}, ..., q_{i_m})$ on vertices $(v_{i_1}, ..., v_{i_m})$. We want to show that M is a cycle that we deliberately added during our construction - one that exists on a complete set of m consecutive vertices of 1s, meaning that $\{v_{i_1}, ..., v_{i_m}\} = VI_l$ for some l.

To do this, we will first show that $(q_{i_1}, ..., q_{i_m}) = (1, ..., 1)$:

Since the degree of each vertex in a (k,n)-cycle is k (Lemma 5.17), we can see that $q_{i_j} = 1$ for each $j \geq k+2$, since otherwise v_{i_j} would have degree at least $k+2$, a contradiction.

Next, we rule out $q_{i_{k+1}} = 0$:

Assume for a contradiction that $q_{i_{k+1}} = 0$, and look at q_{i_k}. If $q_{i_k} = 0$, then we get a $(k, k+1)$-cycle embedded in our $(k, m+1)$-cycle, contradicting Lemma 5.19. Thus, $q_{i_k} = 1$.

Since v_{i_k} must have degree k by Lemma 5.17, and we have currently accounted for only $k-1$ edges, it must form an edge with a vertex past $v_{i_{k+1}}$, call it v_{i_j}. Since we argued earlier that $q_{i_j} \neq 0$, we must have $q_{i_j} = 1$. This gives us that the edge between v_{i_k} and v_{i_j} must be from an intended (k, n)-cycle that we introduced as we went from $\Gamma(PA)$ to $\Gamma(PA)'$, which means that q_{i_k} and q_{i_j} are in a block of consecutive 1s, which contradicts $q_{i_{k+1}} = 0$. Hence, we must have that $q_{i_{k+1}} = 1$.

Now, we can inductively rule out $q_{i_j} = 0$ for $0 \leq j \leq k$:

Suppose $q_{i_k} = 0$. Then, to ensure v_{i_k} has degree k, it must be involved in an edge with a vertex past v_{i_k}, say v_{i_j}. Now, $q_{i_k} = 0$ means that $q_{i_j} = 0$, since it cannot be one of our newly "introduced edges" from a (k, n)-cycle. But, as discussed previously, this cannot be the case, as we only introduce (k, n)-cycles on vertices with value 1.

Repeating this argument gives us that every point before k must have value 1, giving us $(q_{i_1}, ..., q_{i_m}) = (1, ..., 1)$. Thus, all the edges of M are edges we added as (k, n)-cycles. We only added edges between points within some VI_l (consecutive independent points), and so any k vertices only form an edge if they are in the same VI_l. As M is connected, all points of M must be in the same VI_l. Then, we have a (k, m)-cycle embedded in a (k, n)-cycle, which means that (by Lemma 5.19) M is in fact a (k, n)-cycle we intentionally added. This completes the proof.

□

Lemma 5.21. *Suppose that $A = (a_i)_{i\in\mathbb{N}}$ and $B = (b_i)_{i\in\mathbb{N}}$ are different strictly increasing sequences of natural numbers with $a_1, b_1 \geq k+2$. Then $\Gamma(PA)' \ncong \Gamma(PB)'$*

Proof. Since A and B are different sequences, there exists $j \in \mathbb{N}$ such that $a_j \neq b_j$. We can assume without loss of generality that $a_j < b_j$. Hence $\Gamma(PA)'$ embeds a (k, a_j)-cycle, but since $a_j \notin B$, we have by Lemma 5.20 that a (k, a_j)-cycle does not embed in $\Gamma(PB)'$. Hence, $\Gamma(PA)'$ and $\Gamma(PB)'$ have different ages, and thus they are not isomorphic. □

With these results, we can now prove the main aim of this section.

Theorem 5.22. *For each $k \geq 2$, there exists $2^{\aleph_0}$ non-isomorphic MB-homogeneous k-hypergraphs, each of which is bimorphism equivalent to R_k.*

Proof. Since there are $2^{\aleph_0}$ strictly increasing sequences of natural numbers with first entry greater than $k+2$, we have $2^{\aleph_0}$ non-isomorphic k-hypergraphs generated by these sequences using Definition 5.15, due to Lemma 5.21. Furthermore, Lemma 5.16 ensures that each of these examples satisfy $(\therefore)$ and (Δ), and hence by Remark 5.6 and Theorem 5.3 each example is bimorphism equivalent to R_k, and MB-homogeneous. $\square$

Chapter Six

Conclusion and Further Questions

Throughout this work, and in particular the final two chapters, I have presented an initial investigation into homomorphism-homogeneous k-hypergraphs, which have yielded some novel results. However, there are various other avenues which warrant exploring here, which I would be excited to see future progress in.

In Coleman's PhD Thesis [6], after providing uncountably many examples of MB-homogeneous graphs, he conjectured at the complete classification (up to bimorphism) of all MB-homogeneous. This question was answered positively by Aranda and Hartman only two years later [2]. Since Chapter 5 of this work essentially replicates many of Coleman's results for k-hypergraphs, it would be interesting for someone to take up the position of Aranda and Hartman here and try to answer the following question.

Question 6.1. *Up to bimorphism-equivalence, what is the classification of the MB-homogeneous k-hypergraphs?*

Coleman [6] was also able to extend Frucht's Theorem to MB-homogeneous graphs (Theorem 8.2.11 in his Thesis), and I think there is potential to extend this theorem to the k-hypergraph case too. I would be very excited to see this question answered.

Question 6.2. *Does every finite group H arise as the automorphism group of an MB-homogeneous k-hypergraph Γ, for each $k \geq 3$?*

Throughout this dissertation, I also focussed mainly on MY-homogeneity, and within this MB-homogeneity in particular. I would be excited to see attempts at classifying other forms of homomorphism-homogeneous graphs and k-hypergraphs, and investigating some of their properties, in a similar vein to my investigations. This would help to provide a more complete account of homomorphism-homogeneous structures than the partial picture we currently have.

Question 6.3. *For each XY such that $X \in \{I, M, H\}$, $Y \in \{H, E, M, B, I', A\}$, can we classify the XY-homogeneous k-hypergraphs, for each $k \geq 2$? Some of these have been classified, for example for IA-homogeneity when $k = 2$, but this is unanswered in general.*

This is a very ambitious question, and progress on any form of homomorphism-homogeneity for any $k \geq 2$ would present a significant contribution to the field.

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